

L^2 AND H^p BOUNDEDNESS OF STRONGLY SINGULAR OPERATORS AND OSCILLATING OPERATORS ON HEISENBERG GROUPS

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ABSTRACT. In this paper we establish sharp L^2 and H^p boundedness results for strongly singular operators and oscillating operators on Heisenberg groups.

1. INTRODUCTION

The setting of this paper is the Heisenberg group \mathbb{H}_a^n , $a \in \mathbb{R}^*$, realized as \mathbb{R}^{2n+1} equipped with the group law,

$$(x, t) \cdot (y, s) = (x + y, s + t - 2ax^T Jy), \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

This group is equipped with the following anisotropic dilations,

$$\lambda \cdot (x, t) = (\lambda x, \lambda^2 t), \quad \lambda > 0.$$

For $K \in \mathcal{D}'(\mathbb{H}_a^n)$ we denote by T_K the convolution operator defined by K , i.e.,

$$T_K f(x, t) := K * f(x, t) = \int_{\mathbb{H}_a^n} K((x, t) \cdot (y, s)^{-1}) f(y, s) dy dx, \quad f \in C_0^\infty(\mathbb{H}_a^n).$$

We say that the operator T_K is bounded on $L^p(\mathbb{H}^n)$ if there exist a $C > 0$ such that

$$\|T_K f\|_p \leq C \|f\|_p, \quad \text{for all } f \in C_0^\infty(\mathbb{H}_a^n).$$

A natural quasi-norm on the Heisenberg group is given by

$$\rho(x, t) = (|x|^4 + t^2)^{1/4}, \quad (x, t) \in \mathbb{H}_a^n.$$

This quasi-norm satisfies $\rho(\lambda \cdot (x, t)) = \lambda \rho(x, t)$. For this quasi-norm, we define the strongly singular kernels,

$$K_{\alpha, \beta}(x, t) = \rho(x, t)^{-(2n+2+\alpha)} e^{i\rho(x, t)^{-\beta}} \chi(\rho(x, t)), \quad \alpha > 0, \quad \beta > 0,$$

where χ is a smooth bump function in a small neighborhood of the origin. This operator was introduced by Lyall [13] who showed that $T_{K_{\alpha, \beta}}$ is bounded when $\alpha \leq n\beta$. This result was obtained by using the Fourier transform on the Heisenberg group in combination with involved estimates on oscillatory integrals. Subsequently, Laghi-Lyall [10] obtained sharp results in the special case $a^2 < C_\beta$ (where C_β is given by (1.1)) by using a version for the Heisenberg group of the L^2 -boundedness theorem for non-degenerate oscillatory integral operators of Hörmander [9]. In this paper, we shall consider the cases $a^2 \geq C_\beta$ and obtain sharp conditions using the theory for oscillatory integral operators with degenerate phases (see Section 2). Recall that the theory of the degenerate oscillatory integral operators was developed in depth to study X-ray transforms (see, e.g., Greenleaf-Seeger [7]).

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Strongly singular convolution operators were originally considered on \mathbb{R}^n . Such operators correspond to suitable oscillating multipliers. They were first studied, by Fourier transform techniques, in the Euclidean setting with $\rho(x) = |x|$ by Hirschman [8] in the case $d = 1$, and in higher dimensions by Wainger [21], Fefferman [3], and Fefferman-Stein [4].

Similar kind of convolution operators with kernels of the form $\frac{1}{|x|^{n-\alpha}}e^{i|x|^\beta}$, $\alpha, \beta > 0$, were introduced by Sjölin [17, 18, 19]. Such kernels have no singularity near the origin, but they assume relatively small decaying property at infinity. Notice that the case $\beta = 1$ corresponds to the kernel of Bochner-Riesz means. For $\beta \neq 1$, the (L^p, L^q) estimates and Hardy space estimates hold (see Miyachi [14], Pan-Sampson [16] and Sjölin [17, 18, 19]). The difference between the two cases comes from the fact that the phase kernel $|x - y|^\beta$ is degenerate only if $\beta = 1$. In this paper, we also consider the analogous problem on the Heisenberg groups for the following kernels,

$$L_{\alpha,\beta}(x, t) = \rho(x, t)^{-(2n+2-\alpha)}e^{i\rho(x,t)^\beta}\chi(\rho(x, t)^{-1}), \quad \beta > 0.$$

We denote by $T_{L_{\alpha,\beta}}$ the group convolution operators with the kernel $L_{\alpha,\beta}$. In the literature, the operators $T_{K_{\alpha,\beta}}$ (resp., $T_{L_{\alpha,\beta}}$) are called strongly singular operators (resp., oscillating convolution operators).

In the first part of this paper, we shall find the optimal ranges of α and β where the convolution operators associated with $K_{\alpha,\beta}$ and $L_{\alpha,\beta}$ are bounded on $L^2(\mathbb{H}_a^n)$.

For $a^2 \geq C_\beta$, the phase doesn't satisfy the non-degeneracy condition anymore. Therefore, we need to deal with oscillatory integral operators with degenerate phases. A theory for this kind of operators has been developed by considering various conditions on phase functions to give different decaying properties (see [7]). We shall rely on the results of Greenleaf-Seeger [6] and Pan-Sogge [15]. To use such theory we shall carefully investigate the folding type for our phases. Interestingly enough, we have different folding types according to the values of the parameters a and β . Before stating our results, we recall the previous results of Laghi-Lyall [10] and Lyall [13]. Set

$$(1.1) \quad C_\beta = \frac{\beta + 2}{2}(2\beta + 5 + \sqrt{(2\beta + 5)^2 - 9}).$$

Then we have

Theorem (Laghi-Lyall [10], Lyall [13]).

- (1) $T_{K_{\alpha,\beta}}$ is bounded on $L^2(\mathbb{H}_a^n)$ if $\alpha \leq n\beta$.
- (2) If $0 < a^2 < C_\beta$, then $T_{K_{\alpha,\beta}}$ is bounded on $L^2(\mathbb{H}_a^n)$ if and only if $\alpha \leq (n + 1/2)\beta$.

The first main result of this paper gives sharp L^2 boundedness results for $T_{K_{\alpha,\beta}}$ when $a^2 \geq C_\beta$.

Theorem 1.1.

- (1) If $a^2 > C_\beta$, then $T_{K_{\alpha,\beta}}$ is bounded on $L^2(\mathbb{H}_a^n)$ if and only if $\alpha \leq (n + \frac{1}{3})\beta$.
- (2) If $a^2 = C_\beta$, then $T_{K_{\alpha,\beta}}$ is bounded on $L^2(\mathbb{H}_a^n)$ if and only if $\alpha \leq (n + \frac{1}{4})\beta$.

For the operators $T_{L_{\alpha,\beta}}$, we also have the sharp L^2 boundedness results except when $\beta = 1$ and $\beta = 2$.

Theorem 1.2.

- (1) If $0 < \beta < 1$, then $T_{L_{\alpha,\beta}}$ is bounded on L^2 if and only if one of the following condition holds.
 - (i) $a^2 < C_\beta$ and $\alpha \leq (n + \frac{1}{2})\beta$,

- (ii) $a^2 = C_\beta$ and $\alpha \leq (n + \frac{1}{4})\beta$,
- (iii) $a^2 > C_\beta$ and $\alpha \leq (n + \frac{1}{3})\beta$.
- (2) If $1 < \beta < 2$, then $T_{L_{\alpha,\beta}}$ is bounded on L^2 if and only if $\alpha \leq (n + \frac{1}{3})\beta$.
- (3) If $2 < \beta$, then $T_{L_{\alpha,\beta}}$ is bounded on L^2 if and only if $\alpha \leq (n + \frac{1}{2})\beta$.

In [10] Laghi-Lyall reduced the boundedness problem for operators on the Heisenberg group to that for the local operators and used a version of Hörmander's L^2 -boundedness theorem on the Heisenberg group. However, as we shall show, we may view the operators on the Heisenberg group as operators on Euclidean space \mathbb{R}^{2n+1} . This will enable us to use the oscillatory integral estimates of Greenleaf-Seeger [6] and Pan-Sogge [15] on Euclidean space.

For the cases $\beta = 1$ or $\beta = 2$, we also can obtain the sharp results for some value a where the phase becomes non-degenerate or has folds of type 2. However, in these cases, higher order types of folds than 3 appear for some values of a and the degenerate oscillatory integral estimates have not been obtained optimally yet for these cases. The theory have been established optimally only for phases with one or two types of folds (see Greenleaf-Seeger [6] and Pan-Sogge [15]).

For $p > 1$, L^p boundedness can be obtained by interpolation between the L^2 boundedness estimates and some L^1 boundedness estimates for dyadic-piece operator. We refer to Laghi [10, Theorem 5] for the case $a^2 < C_\beta$ except the endpoint. Using this typical interpolation technique, it is also possible to obtain the L^p boundedness in the case $a^2 \geq C_\beta$.

Theorem 1.3.

- (1) If $a^2 > C_\beta$, then $T_{K_{\alpha,\beta}}$ is bounded on $L^p(\mathbb{H}_a^n)$ if $\alpha - (n + \frac{1}{3})\beta < 2\beta(n + \frac{1}{3}) \left| \frac{1}{p} - \frac{1}{2} \right|$.
- (2) If $a^2 = C_\beta$, then $T_{K_{\alpha,\beta}}$ is bounded on $L^p(\mathbb{H}_a^n)$ if $\alpha - (n + \frac{1}{4})\beta < 2\beta(n + \frac{1}{4}) \left| \frac{1}{p} - \frac{1}{2} \right|$.

Theorem 1.4.

- (1) If $0 < \beta < 1$, then $T_{L_{\alpha,\beta}}$ is bounded on $L^p(\mathbb{H}_a^n)$ if one of the following holds.
 - (i) $a^2 < C_\beta$ and $\alpha - (n + \frac{1}{2})\beta < 2\beta(n + \frac{1}{2}) \left| \frac{1}{p} - \frac{1}{2} \right|$,
 - (ii) $a^2 = C_\beta$ and $\alpha - (n + \frac{1}{4})\beta < 2\beta(n + \frac{1}{4}) \left| \frac{1}{p} - \frac{1}{2} \right|$,
 - (iii) $a^2 < C_\beta$ and $\alpha - (n + \frac{1}{3})\beta < 2\beta(n + \frac{1}{3}) \left| \frac{1}{p} - \frac{1}{2} \right|$.
- (2) If $1 < \beta < 2$, then $T_{L_{\alpha,\beta}}$ is bounded on $L^p(\mathbb{H}_a^n)$ if $\alpha - 2(n + \frac{1}{3})\beta < 2\beta(n + \frac{1}{3}) \left| \frac{1}{p} - \frac{1}{2} \right|$.
- (3) If $2 < \beta$, then $T_{L_{\alpha,\beta}}$ is bounded on $L^p(\mathbb{H}_a^n)$ if $\alpha - 2(n + \frac{1}{2})\beta < 2\beta(n + \frac{1}{2}) \left| \frac{1}{p} - \frac{1}{2} \right|$.

In the second part of this paper, we turn our attention to the boundedness on Hardy spaces H^p ($p \leq 1$) of the operators $T_{K_{\alpha,\beta}}$ and $T_{L_{\alpha,\beta}}$.

For the analogous operators on \mathbb{R}^n , the boundedness on Hardy spaces was proved up to the endpoint cases by Sjölin [17, 19]. In this case, the operator can be thought as a multiplier operator $Tf = (m\hat{f})^\vee$ and we have the relation $c_p \sum_{j=1}^n \|R_j f\|_{L^p} \leq \|f\|_{H^p} \leq C_p \sum_{j=1}^n \|R_j f\|_{L^p}$ and we see that derivatives of the symbol $\frac{\xi_j}{|\xi|} m(\xi)$ of the multiplier $R_j m(D)$ are pointwisely bounded by the derivatives of the symbol $m(\xi)$. These things make it possible to calculate the H^p norm accurately to obtain the sharp boundedness result including for the endpoint cases (see Miyachi [14]).

The above outline seems difficult to adapt to the Heisenberg group. Instead we shall rely on the molecular decomposition for Hardy spaces.

Theorem 1.5. *Let $p \in (0, 1)$ and let α and β be real numbers such that $(\frac{1}{p} - 1)(2n + 2)\beta + \alpha < 0$. Then*

- (1) The operator $T_{K_{\alpha,\beta}}$ is bounded on H^p space.
- (2) For $\beta \neq 1$, the operator $T_{L_{\alpha,\beta}}$ is bounded on H^p space.

These conditions are optimal except for the endpoint case $(\frac{1}{p} - 1)(2n + 2)\beta + \alpha = 0$.

This paper is organized as follows. In Section 2, we reduce our problem on the Heisenberg group to a local oscillatory integral estimates on Euclidean space. In Section 3, we recall some essential results for the oscillatory integral operators with degenerate phase functions and study geometry of the canonical relation and projection maps associated with the phase functions of the reduced operators, which will complete the proof of Theorem 1.1 and Theorem 1.2. In section 4, we recall some background on hardy spaces on the Heisenberg group and its basic properties. In section 5, we prove Theorem 1.5. In Section 6, we show that the conditions of Theorem 1.5 are sharp except the endpoint cases.

NOTATION

We will use the notation \lesssim instead of $\leq C$ when the constant C depends only on the fixed parameters such as a, α, β and n . In addition, we will use the notation $A \sim B$ when both inequalities $A \lesssim B$ and $A \gtrsim B$ hold.

2. DYADIC DECOMPOSITION AND LOCALIZATION

In this section we reduce our problems to some oscillatory integral estimates problem on Euclidean space \mathbb{R}^{2n+1} . This reduction is well-known for operators on Euclidean space (see Stein [20]). The issue of this reduction on the Heisenberg group is to control the localized operators $\tilde{T}_j^{k,l}$ in (2.6) uniformly for (g_k, g_l) such that $\rho(g_k \cdot g_l^{-1}) \leq 2$. Note that the cut-off functions $\eta(\rho((x, t) \cdot g_k^{-1})) \eta(\rho((y, s) \cdot g_l^{-1}))$ have no uniform bound for their derivatives. Nevertheless we get the uniformity after a value-preserving change of coordinates (see (2.8)).

We decompose the kernels $K_{\alpha,\beta}$ and $L_{\alpha,\beta}$ as

$$(2.1) \quad K_{\alpha,\beta}(x, t) = \sum_{j=1}^{\infty} K_{\alpha,\beta}^j, \quad K_{\alpha,\beta}^j := \eta(2^j \rho(x, t)) K_{\alpha,\beta}(x, t),$$

and

$$(2.2) \quad L_{\alpha,\beta}(x, t) = \sum_{j=1}^{\infty} L_{\alpha,\beta}^j, \quad L_{\alpha,\beta}^j := \eta(2^{-j} \rho(x, t)) L_{\alpha,\beta}(x, t),$$

where $\eta \in C_0^\infty(\mathbb{R})$ is a bump function supported in $[\frac{1}{2}, 2]$ such that $\sum_{j=0}^{\infty} \eta(2^j r) = 1$ for all $0 < r \leq 1$. For notational convenience, we omit the index α and β from now on.

Set $T_j f = K_{\alpha,\beta}^j * f$ and $S_j f = L_{\alpha,\beta}^j * f$. Then we have

Lemma 2.1. *For each $N \in \mathbb{N}$, there exist constants $C_N > 0$ and $c_\beta > 0$ such that*

$$(2.3) \quad \begin{aligned} \|T_j^* T_{j'}\|_{L^2 \rightarrow L^2} + \|T_j T_{j'}^*\|_{L^2 \rightarrow L^2} &\leq C_N 2^{-\max\{j, j'\}N} \\ \|S_j^* S_{j'}\|_{L^2 \rightarrow L^2} + \|S_j S_{j'}^*\|_{L^2 \rightarrow L^2} &\leq C_N 2^{-\max\{j, j'\}N} \end{aligned}$$

holds for all j and j' satisfying $|j - j'| \geq c_\beta$.

Proof. The proof follows from the integration parts technique in the typical way, so we omit the details. See Lyall [13, Lemma 2.4] where the proof for T_j is given. \square

By Cotlar-Stein Lemma, we only need to show that there is a constant $C > 0$ such that

$$\|T_j\|_{L^2 \rightarrow L^2} + \|S_j\|_{L^2 \rightarrow L^2} \leq C \quad \forall j \in \mathbb{N}.$$

We consider the dilated kernels

$$(2.4) \quad \begin{aligned} \tilde{K}_{\alpha,\beta}^j(x, t) &= K_{\alpha,\beta}^j(2^{-j} \cdot (x, t)) = \eta(\rho(x, t)) 2^{j(Q+\alpha)} \rho(x, t)^{-Q-\alpha} e^{i2^{j\beta} \rho(x, t)^{-\beta}}, \\ \tilde{L}_{\alpha,\beta}^j(x, t) &= L_{\alpha,\beta}^j(2^{-j} \cdot (x, t)) = \eta(\rho(x, t)) 2^{-j(Q-\alpha)} \rho(x, t)^{-Q+\alpha} e^{i2^{j\beta} \rho(x, t)^{\beta}}. \end{aligned}$$

We define \tilde{T}_j and \tilde{S}_j to be the convolution operators with kernels given by $\tilde{K}_{\alpha,\beta}^j$ and $\tilde{L}_{\alpha,\beta}^j$. Set $f_j(x, t) = f(2^{-j} \cdot (x, t))$. Then $K_{\alpha,\beta}^j * f(2^{-j} \cdot (x, t)) = 2^{-jQ} (\tilde{K}_{\alpha,\beta}^j * f_j)(x, t)$, and we have

$$(2.5) \quad \begin{aligned} \|T_j f\|_{L^2} &= \|K_{\alpha,\beta}^j * f(x, t)\|_{L^2} = 2^{-jQ/2} \|K_{\alpha,\beta} * f(2^{-j} \cdot (x, t))\|_{L^2} \\ &\leq 2^{-jQ/2} \cdot 2^{-jQ} \|\tilde{K}_{\alpha,\beta}^j * f_j(x, t)\|_{L^2} \\ &\leq 2^{-jQ/2} \cdot 2^{-jQ} \|\tilde{T}_j\|_{L^2 \rightarrow L^2} \|f_j\|_{L^2} \\ &\leq 2^{-jQ} \|\tilde{T}_j\|_{L^2 \rightarrow L^2} \|f\|_{L^2}. \end{aligned}$$

Similarly, we have $\|S_j f\|_{L^2} \leq 2^{jQ} \|\tilde{S}_j\|_{L^2 \rightarrow L^2} \|f\|_{L^2}$. It follows that it is enough to prove that $\|\tilde{T}_j\|_{L^2 \rightarrow L^2} \lesssim 2^{jQ}$ and $\|\tilde{S}_j\|_{L^2 \rightarrow L^2} \lesssim 2^{-jQ}$.

Now, we further modify our operators to some operators defined locally using the fact that the kernels of \tilde{T}_j and \tilde{S}_j are supported in $\{(x, t) : \rho(x, t) \leq 2\}$. To do this we find a set of point $G = \{g_k : k \in \mathbb{N}\}$ such that $\bigcup_{k \in \mathbb{N}} B(g_k, 2) = \mathbb{H}_a^n$ and each $B(g_k, 4)$ contains only d_n 's other g_l members in G .

We can split $f = \sum_{k=1}^{\infty} f_k$ with each f_k supported in $B(g_k, 2)$. Define

$$(2.6) \quad \tilde{T}_j^{k,l} f(x, t) = \int \tilde{K}_{\alpha,\beta}^j((x, t) \cdot (y, s)^{-1}) \cdot \eta(\rho((x, t) \cdot g_k^{-1})) \eta(\rho((y, s) \cdot g_l^{-1})) f(y, s) dy ds.$$

Then,

$$(2.7) \quad \begin{aligned} \|\tilde{T}_j * f\|_{L^2(\mathbb{H}_a^n)}^2 &\leq \sum_{k=1}^{\infty} \|\tilde{T}_j * f\|_{L^2(B(g_k, 2))}^2 \\ &\leq \sum_{k=1}^{\infty} \|\tilde{T}_j * \sum_{l=1}^{\infty} f_l\|_{L^2(B(g_k, 2))}^2 \\ &\leq \sum_{k=1}^{\infty} \|\tilde{T}_j * \sum_{\{l: \rho(g_l \cdot g_k^{-1}) \leq 2\}} f_l\|_{L^2((B(g_k, 2)))}^2 \\ &\lesssim \sum_{k=1}^{\infty} \sum_{l: \rho(g_l \cdot g_k^{-1}) \leq 2} \|\tilde{T}_j^{k,l}\|_{L^2 \rightarrow L^2} \|f_l\|_{L^2}^2 \\ &\lesssim \sup_{\rho(g_l \cdot g_k^{-1}) \leq 2} \|\tilde{T}_j^{k,l}\|_{L^2 \rightarrow L^2} \|f\|_{L^2}^2. \end{aligned}$$

We note that

$$(2.8) \quad \det(D_{x,t}((x, t) \cdot g)) = 1 \text{ for all } g \in \mathbb{H}_a^n.$$

Then, using the coordinate change $(y, s) \rightarrow ((y, s) \cdot g_k)$ and substituting $(x, t) \rightarrow ((x, t) \cdot g_k)$ in (2.6), we get

$$(2.9) \quad \begin{aligned} &\tilde{T}_j^{k,l} f((x, t) \cdot g_k) \\ &= \int \tilde{K}_{\alpha,\beta}^j((x, t) \cdot (y, s)^{-1}) \eta(\rho(x, t)) \eta(\rho((y, s) \cdot (g_k \cdot g_l^{-1}))) f((y, s) \cdot g_k) dy ds. \end{aligned}$$

Notice that $\rho(g_k \cdot g_l^{-1}) \lesssim 1$. Set $\psi((x, t), (y, s)) = \eta(\rho(x, t))\eta(\rho(y, s) \cdot (g_k \cdot g_l^{-1}))$ and write f just for $f((\cdot) \cdot g_k)$. Then $\sup_{\rho(g_l \cdot g_k^{-1}) \leq 2} \|\tilde{T}_j^{k,l}\|$ will be achieved if we prove $\|\mathcal{T}_j\|_{L^2 \rightarrow L^2} \lesssim 2^{jQ}$ for

$$(2.10) \quad \mathcal{T}_j f(x, t) = \int \tilde{K}_{\alpha, \beta}^j((x, t) \cdot (y, s)^{-1}) \psi((x, t), (y, s)) f(y, s) dy ds$$

with a compactly supported smooth function ψ . Finally we set

$$(2.11) \quad \begin{aligned} A_j(x, t) &= 2^{j\alpha} \mu(x, t) e^{i2^{j\beta} \rho(x, t)^{-\beta}}, \\ B_j(x, t) &= 2^{j\alpha} \mu(x, t) e^{i2^{j\beta} \rho(x, t)^{\beta}}, \end{aligned}$$

where μ is a smooth function supported on the set $\{(x, t) \in \mathbb{R}^{2n+1} : \frac{1}{10} \leq \rho(x, t) \leq 10\}$. We define the operators L_{A_j} and L_{B_j} by

$$(2.12) \quad \begin{aligned} L_{A_j} f(x, t) &= \int A_j((x, t) \cdot (y, s)^{-1}) \psi((x, t), (y, s)) f(y, s) dy ds, \\ L_{B_j} f(x, t) &= \int B_j((x, t) \cdot (y, s)^{-1}) \psi((x, t), (y, s)) f(y, s) dy ds. \end{aligned}$$

We shall deduce Theorem 1.1 and Theorem 1.2 from the following propositions.

Proposition 2.2.

(1) If $a^2 > C_\beta$, then

$$\|L_{A_j}\|_{L^2 \rightarrow L^2} \lesssim 2^{j(\alpha - (n + \frac{1}{3})\beta)}, \quad \forall j \in \mathbb{N}.$$

(2) If $a^2 = C_\beta$, then

$$\|L_{A_j}\|_{L^2 \rightarrow L^2} \lesssim 2^{j(\alpha - (n + \frac{1}{4})\beta)}, \quad \forall j \in \mathbb{N}.$$

Proposition 2.3.

(1) If $0 < \beta < 1$, then,

(i) For $a^2 < C_\beta$,

$$\|L_{B_j}\|_{L^2 \rightarrow L^2} \lesssim 2^{j(\alpha - (n + \frac{1}{2})\beta)} \quad \forall j \in \mathbb{N}.$$

(ii) For $a^2 = C_\beta$,

$$\|L_{B_j}\|_{L^2 \rightarrow L^2} \lesssim 2^{j(\alpha - (n + \frac{1}{4})\beta)} \quad \forall j \in \mathbb{N}.$$

(iii) For $a^2 > C_\beta$,

$$\|L_{B_j}\|_{L^2 \rightarrow L^2} \lesssim 2^{j(\alpha - (n + \frac{1}{3})\beta)} \quad \forall j \in \mathbb{N}.$$

(2) If $1 < \beta < 2$, then

$$\|L_{B_j}\|_{L^2 \rightarrow L^2} \lesssim 2^{j(\alpha - (n + \frac{1}{3})\beta)} \quad \forall j \in \mathbb{N}.$$

(3) If $2 < \beta$, then

$$\|L_{B_j}\|_{L^2 \rightarrow L^2} \lesssim 2^{j(\alpha - (n + \frac{1}{2})\beta)} \quad \forall j \in \mathbb{N}.$$

We get the first main result of this paper assuming these propositions:

Proof of Theorem 1.1 and Theorem 1.2. From the reductions (2.5), (2.7) and (2.9), in order to prove Theorem 1.1 it is enough to prove that $\|\mathcal{T}_j\|_{L^2 \rightarrow L^2} \lesssim 2^{jQ}$ for the operators \mathcal{T}_j given in (2.10). From (2.4) and (2.11) we have $\mathcal{T}_j = 2^{jQ} L_{A_j}$ with a suitable function μ , and so $\|\mathcal{T}_j\|_{L^2 \rightarrow L^2} = 2^{jQ} \|L_{A_j}\|_{L^2 \rightarrow L^2}$. Therefore, the estimates of Proposition 2.2 yield Theorem 1.1. In the same way, Proposition 2.3 establishes Theorem 1.2. \square

Proof of Theorem 1.3 and Theorem 1.4. By the duality argument, it is enough to prove for $p < 2$. We shall prove only the case (1) of Theorem 1.3, the other cases will follow from the same argument. Suppose $p < 2$ and $a^2 > C_\beta$. Since $\|T_j\|_{L^2 \rightarrow L^2} \lesssim \|L_{A_j}\|_{L^2 \rightarrow L^2}$, Proposition 2.2 deduces the following.

$$\|T_j\|_{L^2 \rightarrow L^2} \lesssim 2^{j(\alpha - (n + \frac{1}{3})\beta)}$$

and trivial young's inequality deduces

$$\|T_j\|_{L^1 \rightarrow L^1} \lesssim 2^{j(\alpha)}.$$

By interpolation we obtain

$$\|T_j\|_{L^p \rightarrow L^p} \lesssim 2^{j(\alpha 2(\frac{1}{p} - \frac{1}{2}) + (\alpha - (n + \frac{1}{3})\beta)2(1 - \frac{1}{p}))} = 2^{j(\alpha - 2(n + \frac{1}{3})(1 - \frac{1}{p}))}.$$

We can sum the geometric series if $\alpha - 2(n + \frac{1}{3})(1 - \frac{1}{p})$. This completes the proof. \square

In the next section, we shall briefly review on the theory related to the operators L_{A_j} and L_{B_j} . We will make use of geometric properties of the phase function $\rho(x, t)^\beta$ to prove Proposition 2.2 and Proposition 2.3.

3. L^2 ESTIMATES

We begin with the $L^2 \rightarrow L^2$ theory for oscillatory integral operators. The operators we are concern with are of the form

$$T_\lambda^\phi f(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x, y)} a(x, y) f(y) dy,$$

where $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Suppose that the phase function ϕ satisfies $\det\left(\frac{\partial^2 \phi}{\partial x_i \partial y_j}\right) \neq 0$ on the support of a , we say that ϕ is non-degenerate. We say that ϕ is degenerate if there is some point (x_0, y_0) where $\det\left(\frac{\partial^2 \phi}{\partial x_i \partial y_j}\right)\Big|_{(x_0, y_0)}$ equals to zero. For non-degenerate phases, we have the fundamental theorem of Hörmander.

Theorem 3.1 (Hörmander [9]). *Suppose that the phase function ϕ is non-degenerate. Then we have*

$$\|T_\lambda^\phi\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-\frac{n}{2}} \quad \forall \lambda \in [1, \infty).$$

This theorem gives sharp decaying rate of the norm $\|T_\lambda^\phi\|_{L^2 \rightarrow L^2}$ in terms of λ . However, the phase functions of our operators L_{A_j} and L_{B_j} can become degenerate according to the values of a and β (see Lemma 3.4 and Lemma 3.5). For a degenerate phase function ϕ , the optimal number κ_ϕ for which the inequality $\|T_\lambda\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-\kappa_\phi}$ holds would be less than $\frac{n}{2}$. The number κ_ϕ 's are related to the type of fold of the phase ϕ (see Definition 3.2). For phases whose types of folds are ≤ 3 , the sharp numbers κ_ϕ were obtained by Greenleaf-Seeger [6] and Pan-Sogge [15]. We shall use the results. The sharp results for folding types ≤ 3 in [6] are the best known results and there are no optimal results for folding types > 3 except some special cases established by Cuccagna [2].

It is well-known that the decaying property is strongly related to the geometry of the canonical relation,

$$(3.1) \quad C_\phi = \{(x, \partial_x \phi(x, y), y, -\partial_y \phi(x, y)) ; x, y \in \mathbb{R}^n\} \subset T^*(\mathbb{R}_x^n) \times T^*(\mathbb{R}_y^n).$$

Definition 3.2. Let M_1 and M_2 be smooth manifolds of dimension n , and let $f : M_1 \rightarrow M_2$ be a smooth map of corank ≤ 1 . Let $S = \{P \in M_1 : \text{rank}(Df) < n \text{ at } P\}$ be the singular set of f . Then we say that f has a k -type fold at a point $P_0 \in S$ if

- (1) $\text{rank}(Df)|_{P_0} = n - 1$,
- (2) $\det(Df)$ vanishes of k order in the null direction at P_0 .

Here, the null direction is the unique direction vector v such that $(D_v f)|_{P_0} = 0$.

Now we consider the two projection maps

$$(3.2) \quad \pi_L : C_\Phi \rightarrow T^*(\mathbb{R}_x^n) \quad \text{and} \quad \pi_R : C_\Phi \rightarrow T^*(\mathbb{R}_y^n).$$

Proposition 3.3 ([6],[15]). *Suppose that the projection maps π_L and π_R have 1-type folds (Whitney folds) singularities, then*

$$\|T_\lambda f\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-\frac{(n-1)}{2}-\frac{1}{3}} \|f\|_{L^2(\mathbb{R}^n)} \quad \forall \lambda \in [1, \infty).$$

If the projection maps π_L and π_R have 2-type folds singularities, then

$$\|T_\lambda f\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-\frac{(n-1)}{2}-\frac{1}{4}} \|f\|_{L^2(\mathbb{R}^n)} \quad \forall \lambda \in [1, \infty).$$

In order to use Proposition 3.3, we shall study the projection maps (3.2) associated to the phase function of the operators L_{A_j} and L_{B_j} . Recall that $\rho(x, t) = (|x|^4 + t^2)^{1/4}$ and the phase function ϕ of the integral operators L_{A_j} and L_{B_j} is

$$\phi(x, t, y, s) = \rho^{-\beta} \left((x, t) \cdot (y, s)^{-1} \right).$$

To write the group law explicitly, we write $x = (x^1, x^2)$ and $y = (y^1, y^2)$ with $x^j, y^j \in \mathbb{R}^n$. Set $\Phi(x, t) = \rho(x, t)^{-\beta}$. Then

$$(3.3) \quad \phi(x, t, y, s) = \Phi \left(x^1 - y^1, x^2 - y^2, t - s - 2a(x^1 y^2 - x^2 y^1) \right).$$

For notational purpose set $t = x_{2n+1}$ and $s = y_{2n+1}$. To determine whether the phase function Φ is non-degenerate, we need to calculate the determinant of the matrix,

$$H = \left(\frac{\partial^2 \phi(x, t, y, s)}{\partial y_i \partial x_j} \right).$$

The determinant is calculated in Laghi [10]. However we give a somewhat simpler computation by considering the matrix L associated naturally with the matrix H (see below), which will also be useful in Lemma 3.6 and the proof of Proposition 2.2 and Proposition 2.3.

For simplicity, we write $(\mathbf{x}, \mathbf{t}) = (x, t) \cdot (y, s)^{-1}$. By the Chain Rule, for $1 \leq i, j \leq n$, we have

$$(3.4) \quad \begin{aligned} \frac{\partial}{\partial x_j} \phi(x, t, y, s) &= [\partial_j + 2ay_{n+j} \partial_{2n+1}] \Phi(\mathbf{x}, \mathbf{t}), \\ \frac{\partial}{\partial x_{j+n}} \phi(x, t, y, s) &= [\partial_{j+n} - 2ay_j \partial_{2n+1}] \Phi(\mathbf{x}, \mathbf{t}). \end{aligned}$$

Using the Chain Rule once more, we get

$$\begin{aligned}
(3.5) \quad & \frac{\partial}{\partial y_i} \frac{\partial}{\partial x_j} \phi(x, t, y, s) = [(\partial_i + 2ax_{n+i} \partial_{2n+1})(\partial_j + 2ay_{n+j} \partial_{2n+1})] \Phi(\mathbf{x}, \mathbf{t}), \\
& \frac{\partial}{\partial y_{n+i}} \frac{\partial}{\partial x_j} \phi(x, t, y, s) = [(\partial_{n+i} - 2ax_i \partial_{2n+1})(\partial_j + 2ay_{n+j} \partial_{2n+1})] \Phi(\mathbf{x}, \mathbf{t}) + [2a\delta_{ij} \partial_{2n+1}] \Phi(\mathbf{x}, \mathbf{t}), \\
& \frac{\partial}{\partial y_i} \frac{\partial}{\partial x_{n+j}} \phi(x, t, y, s) = [(\partial_i + 2ax_{n+i} \partial_{2n+1})(\partial_{n+j} - 2ay_j \partial_{2n+1})] \Phi(\mathbf{x}, \mathbf{t}) - [2a\delta_{ij} \partial_{2n+1}] \Phi(\mathbf{x}, \mathbf{t}), \\
& \frac{\partial}{\partial y_{n+i}} \frac{\partial}{\partial x_{n+j}} \phi(x, t, y, s) = [(\partial_{n+i} - 2ax_i \partial_{2n+1})(\partial_{n+j} - 2ay_j \partial_{2n+1})] \Phi(\mathbf{x}, \mathbf{t}).
\end{aligned}$$

Define

$$A_a(y) = \begin{pmatrix} I & 2aJy \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then we have

$$\begin{aligned}
(3.6) \quad H(x, t, y, s) &= A_a(x) (\partial_i \partial_j \Phi)(\mathbf{x}, \mathbf{t}) A_a(y)^T + 2a(\partial_{2n+1} \Phi)(\mathbf{x}, \mathbf{t}) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \\
&= A_a(x) \left[(\partial_i \partial_j \Phi) + 2a(\partial_{2n+1} \Phi) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \right] (\mathbf{x}, \mathbf{t}) A_a(y)^T,
\end{aligned}$$

where the second equality holds because $A_a(x) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} A_a(y)^T = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$. Set

$$(3.7) \quad L(x, t, y, s) = \left[(\partial_i \partial_j \Phi) + 2a(\partial_{2n+1} \Phi) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \right] (\mathbf{x}, \mathbf{t}).$$

Thus, to study the matrix H , it is enough to analyze the matrix L . Moreover we have $\det(A_a(x)) = \det(A_a(y)) = 1$ and it implies that $\det(H(x, t, y, s)) = \det(L(x, t, y, s))$. Therefore it is enough to calculate the determinant of L .

To find (3.7) we calculate the Hessian matrix of Φ . For $1 \leq i, j \leq 2n$,

$$\begin{aligned}
(3.8) \quad \partial_j \Phi(\mathbf{x}, \mathbf{t}) &= -\frac{\beta}{4}(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1} (4\mathbf{x}_j |\mathbf{x}|^2), \\
\partial_{2n+1} \Phi(\mathbf{x}, \mathbf{t}) &= -\frac{\beta}{4}(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1} (2\mathbf{t}),
\end{aligned}$$

and

$$\begin{aligned}
(3.9) \quad \partial_i \partial_j \Phi(\mathbf{x}, \mathbf{t}) &= \beta(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-2} [(\beta + 4)|\mathbf{x}|^4 - 2(|\mathbf{x}|^4 + \mathbf{t}^2)] \mathbf{x}_i \mathbf{x}_j - \beta(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1} \delta_{ij} |\mathbf{x}|^2, \\
\partial_i \partial_{2n+1} \Phi(\mathbf{x}, \mathbf{t}) &= \beta(\beta + 4)(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-2} |\mathbf{x}|^2 \mathbf{x}_i \cdot \frac{\mathbf{t}}{2}, \\
\partial_{2n+1}^2 \Phi(\mathbf{x}, \mathbf{t}) &= \beta(\beta + 4)(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-2} \frac{\mathbf{t}}{2} \cdot \frac{\mathbf{t}}{2} - \beta(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1} \frac{1}{2}.
\end{aligned}$$

Set $D = (|\mathbf{x}|^2 \mathbf{x}, \frac{\mathbf{t}}{2})^T$. Then the above computations show that

$$\begin{aligned}
(3.11) \quad & \left[(\partial_i \partial_j \Phi) + 2a(\partial_{2n+1} \Phi) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \right] (\mathbf{x}, \mathbf{t}) \\
&= \beta(\beta + 4)(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-2} D \cdot D^T - \beta(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1} \begin{pmatrix} |\mathbf{x}|^2 I + a\mathbf{t}J + 2\mathbf{x} \cdot \mathbf{x}^T & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\
&= -\beta(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1} (E + R),
\end{aligned}$$

where we set

$$(3.12) \quad B = |\mathbf{x}|^2 I + atJ, \quad K = \mathbf{x} \cdot \mathbf{x}^T, \quad E = \begin{pmatrix} B + 2K & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad R = -\frac{(\beta + 4)}{|\mathbf{x}|^4 + \mathbf{t}^2} D \cdot D^T.$$

Then, from (3.7) and (3.11) we get

$$(3.13) \quad L(x, t, y, s) = [-\beta(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1}(E + R)](\mathbf{x}, \mathbf{t}).$$

Lemma 3.4. *We have*

$$\det H(x, t, y, s) = F((x, t) \cdot (y, s)^{-1}),$$

where $F(x, t) = c_{a,\beta}(|x|^4 + a^2 t^2)^{m_1}(|x|^4 + t^2)^{m_2} f(x, t)$ for some $m_1, m_2, c_{a,\beta} \in \mathbb{R}$ and $f(x, t) = 2(\beta + 1)|x|^8 + (3(\beta + 2) - 2a^2)|x|^4 t^2 + (\beta + 2)a^2 t^4$.

Proof. We write $(\mathbf{x}, \mathbf{t}) = (x, t) \cdot (y, s)^{-1}$ again. In view of (3.6), (3.7) and (3.13), it is enough to show that

$$\det[-\beta(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1}(E + R)] = F(\mathbf{x}, \mathbf{t}).$$

Considering the form of the function F given, we only need to compute $\det(E + R)$. From (3.12) we have

$$E + R = \begin{pmatrix} B + 2K & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \frac{(\beta + 4)}{|\mathbf{x}|^4 + \mathbf{t}^2} D \cdot D^T.$$

For notational convenience, we shall use lower-case letters f_1, \dots, f_m to denote the rows of a given $m \times m$ matrix F . Notice that DD^T is of rank 1 and we have the following equality

$$(3.14) \quad \det(P + Q) = \det(P) + \sum_{j=1}^m \det(p_1^T, \dots, p_{j-1}^T, q_j^T, p_{j+1}^T, \dots, p_m^T),$$

for any $m \times m$ matrices P and Q with $\text{rank } Q = 1$. Recall that $B = |\mathbf{x}|^2 I + atJ$ and $K = \mathbf{x} \cdot \mathbf{x}^T$, then direct calculations show that

$$(3.15) \quad \det(B) = (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^n$$

and

$$(3.16) \quad \begin{aligned} & \sum_{j=1}^n \mathbf{x}_j \det(b_1^T, \dots, b_{j-1}^T, k_j^T, b_{j+1}^T, \dots, b_{2n}^T) + \sum_{j=1}^n \mathbf{x}_{j+n} \det(b_1^T, \dots, b_{j+n-1}^T, k_{j+n}^T, b_{j+n+1}^T, \dots, b_{2n}^T) \\ &= \sum_{j=1}^n \mathbf{x}_j (|\mathbf{x}|^2 \mathbf{x}_j + \mathbf{x}_{n+j} at) (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^{n-1} + \sum_{j=1}^n \mathbf{x}_{j+n} (|\mathbf{x}|^2 \mathbf{x}_{j+n} - \mathbf{x}_j at) (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^{n-1} \\ &= (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^{n-1} |\mathbf{x}|^4. \end{aligned}$$

Thus, from (3.14), (3.15) and (3.16), we get

$$(3.17) \quad \begin{aligned} \det(B + 2K) &= (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^n + 2|\mathbf{x}|^4 (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^{n-1} \\ &= (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^{n-1} (3|\mathbf{x}|^4 + a^2 \mathbf{t}^2). \end{aligned}$$

Using (3.14) once again, we obtain

$$(3.18) \quad \det(E + R) = \det(E) + \frac{1}{2} \sum_{j=1}^{2n} \det \begin{pmatrix} e_1 \\ \vdots \\ e_{j-1} \\ r_j \\ e_{j+1} \\ \vdots \\ e_{2n} \end{pmatrix} + \det \begin{pmatrix} e_1 \\ \vdots \\ e_{2n} \\ r_{2n+1} \end{pmatrix} \\ =: S_1 + S_2 + S_3.$$

From (3.17) we have

$$S_1 = \det \begin{pmatrix} B + 2K & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \det(B + 2K) = \frac{1}{2} (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^{n-1} (3|\mathbf{x}|^4 + a^2 \mathbf{t}^2).$$

Using rank $K = 1$ we get

$$\det \begin{pmatrix} e_1 \\ \vdots \\ e_{j-1} \\ r_j \\ e_{j+1} \\ \vdots \\ e_{2n} \end{pmatrix} = \det \begin{pmatrix} b_1 + 2k_1 \\ \vdots \\ b_{j-1} + 2k_{j-1} \\ \frac{-(\beta+4)|\mathbf{x}|^4}{|\mathbf{x}|^4 + \mathbf{t}^2} k_j \\ b_{j+1} + 2k_{j+1} \\ \vdots \\ b_{2n} + 2k_{2n} \end{pmatrix} = -\frac{(\beta+4)|\mathbf{x}|^4}{|\mathbf{x}|^4 + \mathbf{t}^2} \det \begin{pmatrix} b_1 \\ \vdots \\ b_{j-1} \\ k_j \\ b_{j+1} \\ \vdots \\ b_{2n} \end{pmatrix}.$$

Therefore,

$$S_2 = -\frac{1}{2} \left(\frac{(\beta+4)|\mathbf{x}|^4}{|\mathbf{x}|^4 + \mathbf{t}^2} \right) |\mathbf{x}|^4 (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^{n-1}.$$

Finally,

$$(3.19) \quad S_3 = \det \begin{pmatrix} B + 2K & 0 \\ * & -\frac{\beta+4}{|\mathbf{x}|^4 + \mathbf{t}^2} \frac{\mathbf{t}^2}{4} \end{pmatrix} = -\frac{\beta+4}{|\mathbf{x}|^4 + \mathbf{t}^2} \frac{\mathbf{t}^2}{4} \det(B + 2K) \\ = -\frac{\beta+4}{|\mathbf{x}|^4 + \mathbf{t}^2} \frac{\mathbf{t}^2}{4} (|\mathbf{x}|^4 + a^2 \mathbf{t}^2)^{n-1} (3|\mathbf{x}|^4 + a^2 \mathbf{t}^2).$$

Adding all these terms together, we get

$$\det(E + R) = p(|\mathbf{x}|^4 + a^2 \mathbf{t}^2) q(|\mathbf{x}|^4 + \mathbf{t}^2) f(\mathbf{x}, \mathbf{t}),$$

where $p(r) = c_p r^{m_1}$, $q(r) = r^{m_2}$ for some $m_1, m_2, c_p \in \mathbb{R}$ and

$$f(\mathbf{x}, \mathbf{t}) = 2(\beta+1)|\mathbf{x}|^8 + (3(\beta+2) - 2a^2)|\mathbf{x}|^4 \mathbf{t}^2 + (\beta+2)a^2 \mathbf{t}^4.$$

The proof is complete. \square

Now, we should determine when the determinant of $H(x, t, y, s)$ can be zero for some values (x, t, y, s) with $\rho((x, t) \cdot (y, s)^{-1}) \sim 1$. Furthermore, to determine the type of folds in the degenerate cases, it is crucial to know the shape of the factorization.

Lemma 3.5. *There are nonzero constants γ, c, c_1, c_2, c_3 with $c_1 \neq c_2$ and $c_3 > 0$ that are determined by β and a such that:*

- *Case 1:*
 - If $\beta \in (-1, 0) \cup (0, \infty)$ and $a^2 < C_\beta$, then $f(x, t) > 0$.
 - If $\beta \in (-1, 0) \cup (0, \infty)$ and $a^2 = C_\beta$, then $f(x, t) = \gamma(|x|^2 - ct^2)^2$.
 - If $\beta \in (-1, 0) \cup (0, \infty)$ and $a^2 > C_\beta$, then $f(x, t) = \gamma(|x|^2 - c_1t)(|x|^2 + c_1t)(|x|^2 - c_2t)(|x|^2 + c_2t)$.
- *Case 2:*
 - If $\beta \in (-2, -1)$, then $f(x, t) = \gamma(|x|^2 - c_1t)(|x|^2 + c_1t)(|x|^4 + c_3t^2)$.
- *Case 3:*
 - If $\beta \in (\infty, -2)$, then $f(x, t) < 0$.

Proof. Let $g(y, s) = 2(\beta + 1)y^2 + (3(\beta + 2) - 2a^2)ys + (\beta + 2)a^2s^2$. Then $f(x, t) = g(|x|^4, t^2)$. Suppose $\beta \in (-1, 0) \cup (0, \infty)$. First, we see that $f(x, t) > 0$ for $3(\beta + 2) - 2a^2 > 0$. Secondly, we have $f(x, t) > 0$ if

$$\Delta := 4a^4 - 4(\beta + 2)(2\beta + 5)a^2 + 9(\beta + 2)^2 < 0.$$

This holds if and only if

$$C_\beta^- < a^2 < C_\beta^+,$$

where

$$C_\beta^\pm = \frac{\beta + 2}{2} \left(2\beta + 5 \pm \sqrt{(2\beta + 5)^2 - 9} \right).$$

Observe that

$$\begin{aligned} (3.20) C_\beta^- &= \frac{(\beta + 2)}{2} (2\beta + 5 - \sqrt{(2\beta + 5)^2 - 9}) = \frac{(\beta + 2)}{2} (2\beta + 5 - \sqrt{(2\beta + 2)(2\beta + 8)}) \\ &< \frac{(\beta + 2)}{2} (2\beta + 5 - \sqrt{(2\beta + 2)^2}) = \frac{3(\beta + 2)}{2}. \end{aligned}$$

We can combine the above two conditions as $g(y, s) > 0$ for $a^2 < C_\beta^+$. For $a^2 = C_\beta$, we have $g(y, s) = \gamma(y - cs)^2$ for some $c > 0$. For $a^2 > C_\beta$, we have $g(y, s) = \gamma(y - c_1s)(y - c_2s)$ for some $c_1, c_2 > 0$ since $2(\beta + 1) \cdot (\beta + 2)a^2 > 0$.

Finally, if $\beta \in (-2, -1)$, then $2(\beta + 1)(\beta + 2)a^2 < 0$, and so $g(y, s) = \gamma(y - c_1s)(y + c_2s)$. If $\beta \in (-\infty, -2)$, then $2(\beta + 1) < 0$, $3(\beta + 2) - 2a^2 < 0$ and $\beta + 2 < 0$. Thus $g(y, s) < 0$. This completes the proof. \square

Lemma 3.6. *Let $L_1(x, t, y, s)$ be the upper left $(2n) \times (2n)$ block matrix of $L(x, t, y, s)$ and suppose that (x, t, y, s) is contained in S . If $\beta \neq -4$, then*

$$\det L_1(x, t, y, s) \neq 0.$$

Proof. For simplicity, set $(z, w) := (x, t) \cdot (y, s)^{-1}$. In view of (3.12) and (3.13), except the nonzero common factors, we only need to check that the determinant of

$$M(z, w) = \left(|z|^2 I + awJ + 2z \cdot z^T - (\beta + 4) \frac{|z|^4}{|z|^4 + w^2} x \cdot z^T \right),$$

is nonzero for $(z, w) \neq (0, 0)$. This determinant can be calculated in the same way as the determinant of L by using (3.15) and (3.17). We find

$$\begin{aligned} (3.21) \quad \det(M(z, w)) &= (|z|^4 + a^2 w^2)^n + (|z|^4 + a^2 w^2)^{n-1} |z|^4 \left(2 - (\beta + 4) \frac{|z|^4}{|z|^4 + w^2} \right) \\ &= \frac{(|z|^4 + a^2 w^2)^{n-1}}{|z|^4 + w^2} [-(\beta + 1)|z|^8 + (a^2 + 3)|z|^4 w^2 + a^2 w^4]. \end{aligned}$$

Notice that (z, w) is in S and satisfies

$$(3.22) \quad 2(\beta + 1)|z|^8 + (3(\beta + 2) - 2a^2)|z|^4w^2 + (\beta + 2)a^2w^4 = 0.$$

From (3.21) and (3.22) we get

$$\det(M(z, w)) = \frac{(|z|^4 + a^2w^2)^{n-1}w^2}{|z|^4 + w^2} \frac{w^2}{2}(\beta + 4) [3|z|^4 + a^2w^2].$$

If $w = 0$, then z becomes zero in (3.22). Because $(z, w) \neq (0, 0)$, w should be nonzero. Thus $\det(M(z, w)) \neq 0$. The Lemma is proved. \square

We are now ready to prove our first main theorems by studying the canonical relation (3.1) associated to the phase Φ ,

$$C_\Phi = \{((x, t), \Phi_{(x, t)}, (y, s), -\Phi_{(y, s)})\} \subset T^*(\mathbb{R}^{2n+1}) \times T^*(\mathbb{R}^{2n+1}),$$

and the associated projection maps $\pi_L : C_\Phi \rightarrow T^*(\mathbb{R}^{2n+1})$ and $\pi_R : C_\Phi \rightarrow T^*(\mathbb{R}^{2n+1})$.

Proof of Proposition 2.2 Proposition 2.3. Let

$$S = \{(x, t, y, s) : \det H(x, t, y, s) = 0\}.$$

In view of Proposition 3.3, it is enough to show that on the hypersurface S ,

- (1) If $\beta \in (-2, -1)$ or $\beta \in (-1, 0) \cup (0, \infty)$ and $a^2 > C_\beta$, then both projections π_L and π_R have 1-type folds singularities.
- (2) If $\beta \in (-1, 0) \cup (0, \infty)$ and $a^2 = C_\beta$, then both π_L and π_R have 2-type folds singularities.

We will only prove (1). The second case can be proved in the same way, the only difference is the form of factorizations in Lemma 3.5 which determine the order of types. We need to show that on the hypersurface S , both π_L and π_R have 1-type folds singularities. Recall from Lemma 3.4 that S is a subset of \mathbb{R}^{2n+1} consisting of $(x, t, y, s) \in \mathbb{R}^{2(2n+1)}$ such that

$$F((x, t) \cdot (y, s)^{-1}) = F(x - y, s - t + 2ax^T Jy) = 0 \quad \text{and} \quad \rho((x, t) \cdot (y, s)^{-1}) \sim 1.$$

From the form of F and the fact that $((x, t) \cdot (y, s)^{-1}) \neq 0$, we have

$$S = \{(x, t, y, s) \in \mathbb{R}^{2(2n+1)} \mid f(x - y, s - t + 2ax^T Jy) = 0, \quad \rho((x, t) \cdot (y, s)^{-1}) \sim 1\}.$$

From Theorem 3.5, we have

$$f(x, t) = \gamma(|x|^2 - c_1 t)(|x|^2 + c_1 t)(|x|^2 - c_2 t)(|x|^2 + c_2 t).$$

for some two different constants $c_1, c_2 > 0$.

Note that Lemma 3.6 implies the condition (1) of Definition 3.2 is satisfied. Therefore, it is enough to show the second condition, i.e., at each point $P_0 \in S$ the determinant of Df vanishes with order 1 in each null direction of $d\pi_L$ and $d\pi_R$ at P_0 . Fix a point $P_0 = (x, t, y, s) \in \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$ and assume that P_0 is contained in

$$S_1 =: \{(x, t, y, s) \in \mathbb{R}^{2(2n+1)} \mid |x - y|^2 - c_1(s - t + 2ax^T Jy) = 0\}.$$

We may identify $C_\Phi = \{((x, t), \Phi_{(x, t)}, (y, s), -\Phi_{(y, s)})\}$ with an open set in $\mathbb{R}^{(2n+1)} \times \mathbb{R}^{(2n+1)}$ by the diffeomorphism $\psi : \mathbb{R}^{(2n+1)} \times \mathbb{R}^{(2n+1)} \rightarrow S$ given by

$$\psi(x, t, y, s) = ((x, t), \Phi_{(x, t)}, (y, s), -\Phi_{(y, s)}).$$

Let $v_L \in \mathbb{R}^{2(2n+1)}$ be a null direction of $d\pi_L$ at P_0 , i.e.,

$$\begin{pmatrix} I & 0 \\ \frac{\partial^2 \Phi}{\partial_{(x,t)} \partial_{(x,t)}} & \frac{\partial^2 \Phi}{\partial_{(y,s)} \partial_{(x,t)}} \end{pmatrix} v_L^T = 0.$$

Thus, v_L is of the form $v_L = (0, 0, z, w)$ with $w \in \mathbb{R}^{2n}$ and $s \in \mathbb{R}$ such that

$$(3.23) \quad \frac{\partial^2 \Phi}{\partial_{(y,s)} \partial_{(x,t)}} \begin{pmatrix} z^T \\ w \end{pmatrix} = 0.$$

To check that $\det H(x, t, y, s)$ vanishes of order 1 in the direction v_L , it is enough to show that v_L is not orthogonal to the gradient vector v_g of $\det H(x, t, y, s)$ at P_0 . By a direct calculation we see that the gradient vector v_g is equal to

$$D_{(x,t),(y,s)} \Phi((x, t) \cdot (y, s)^{-1})|_p = (2(x - y) - 2ac_1 a J y, -c_1, -2(x - y) - 2ac_1 x^T J, c_1)$$

Suppose with a view to contradiction that v_L and v_g are orthogonal. It means that

$$(3.24) \quad -2(x - y) \cdot z - 2ac_1 x^T J \cdot z + c_1 w = 0.$$

From (3.6), we have

$$(3.25) \quad \frac{\partial^2 \Phi}{\partial_{(y,s)} \partial_{(x,t)}} \begin{pmatrix} z^T \\ w \end{pmatrix} = A_a(y) \left[(\partial_i \partial_j \Phi) - 2a(\partial_{2n+1} \Phi) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \right] (\mathbf{x}, \mathbf{t}) A_a(x)^T \cdot \begin{pmatrix} z^T \\ w \end{pmatrix}.$$

A simple calculation shows that

$$\begin{aligned} A_a(x)^T \cdot \begin{pmatrix} z^T \\ w \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 2ax_{n+1} & \cdots & -2ax_1 & \cdots & -2ax_n & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n} \\ w \end{pmatrix} \\ &= \left(z_1, z_2, \dots, z_{2n}, 2a(x_{n+1}z_1 + \cdots + x_{2n}z_n - x_1z_{n+1} - \cdots - x_nz_{2n}) + w \right)^T. \end{aligned}$$

On the other hand, from the orthogonal assumption (3.24) we get

$$2a(x_{n+1}z_1 + \cdots - x_nz_{2n}) + w = \frac{2(x - y) \cdot z}{c_1}.$$

Thus,

$$A_a(x)^T \cdot \begin{pmatrix} z^T \\ w \end{pmatrix} = \left(z_1, z_2, \dots, z_{2n}, \frac{2(x-y) \cdot z}{c_1} \right)^T.$$

Recall that

$$\left[(\partial_i \partial_j \Phi) - 2a(\partial_{2n+1} \Phi) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \right] (x, t) = (\beta + 4) \begin{pmatrix} |x|^4 x_1^2 & \cdots & |x|^4 x_1 x_n & |x|^2 x_1 \frac{t}{2} \\ \vdots & \ddots & \vdots & \vdots \\ |x|^4 x_n x_1 & \cdots & |x|^4 x_n^2 & |x|^2 x_n \frac{t}{2} \\ |x|^2 \frac{t}{2} x_1 & \cdots & |x|^2 \frac{t}{2} x_n & \frac{t^2}{4} \end{pmatrix} - (|x|^4 + t^2) \begin{pmatrix} J & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Substituting $x - y$ for x and $t - s + 2ax^T Jy = \frac{|x-y|^2}{c_1}$ for t , where the equality holds since the point P_0 is on the surface S_1 . Then, from $(2n+1)$ -th equality in (3.23) with (3.25), we have

$$(\beta + 4) \left[|x - y|^2 \cdot \frac{1}{2} \frac{|x - y|^2}{c_{\beta,1}} (x - y) \cdot z + \frac{|x - y|^4}{c_{\beta,1}^2} \cdot \frac{2}{c_{\beta,1}} (x - y) \cdot z \right] - \frac{1}{2} (|x - y|^4 + \frac{|x - y|^4}{c_{\beta,1}^2}) \frac{2}{c_{\beta,1}} (x - y) \cdot z = 0.$$

Rearranging it, we obtain

$$\left[\frac{\beta + 2}{2c_{\beta,1}} + \frac{1}{c_{\beta,1}^3} \right] |x - y|^4 (x - y) \cdot z = 0.$$

Thus $(x - y) \cdot z = 0$, and hence

$$A_a(x)^T \cdot \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z_1, z_2, \dots, z_{2n}, 0 \end{pmatrix}^T \quad \text{and} \quad L_1(x, t, y, s) \cdot \begin{pmatrix} z_1, z_2, \dots, z_{2n} \end{pmatrix}^T = 0.$$

Now from $\det L_1 \neq 0$ in Lemma 3.6 we have $z = 0$ and so $w = 0$ from (3.24). This is a contradiction since v_L should be a nonzero direction vector. Therefore v_L and v_R can not be orthogonal.

Now we shall prove the same conclusion for $d\pi_R$ without repeating the calculations. Note that the above argument for $d\pi_L$ is exactly to show that there is no nontrivial solution (z, w) of the system of equation $S(a, x, y)$:

$$\left(\frac{\partial^2}{\partial x_i \partial y_j} \Phi \right) \begin{pmatrix} z^T \\ w \end{pmatrix} = A_a(y) \left[(\partial_i \partial_j \Phi) - 2a(\partial_{2n+1} \Phi) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \right] A_a(x)^T \cdot \begin{pmatrix} z^T \\ w \end{pmatrix} = 0,$$

and

$$(-2(x - y) - 2ac_{\beta,1}x^T J, c_{\beta,1}) \cdot (z, w) = 0.$$

On the other hand, to show the folding type condition for the projection π_R , it is enough to show that there is no nontrivial solution $v_R = (z_0, w_0, 0, 0)$ which satisfies the system of equations :

$$\left(\frac{\partial^2}{\partial y_i \partial x_j} \Phi \right) \begin{pmatrix} z_0^T \\ w_0 \end{pmatrix} = A_a(x) \left[(\partial_i \partial_j \Phi) + 2a(\partial_{2n+1} \Phi) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \right] A_a(y)^T \cdot \begin{pmatrix} z_0^T \\ w_0 \end{pmatrix} = 0,$$

and

$$(2(x - y) + 2ac_{\beta,1}y^T J, -c_{\beta,1}) \cdot (z_0, w_0) = 0.$$

Because $A_{-a}(-x) = A_a(x)$ and $A_{-a}(-y) = A_a(y)$, the above system can be written as follows.

$$\left(\frac{\partial^2}{\partial y_i \partial x_j} \Phi \right) \begin{pmatrix} z_0^T \\ w_0 \end{pmatrix} = A_{-a}(-x) \left[(\partial_i \partial_j \Phi) - 2(-a)(\partial_{2n+1} \Phi) \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \right] A_{-a}(-y)^T \cdot \begin{pmatrix} z_0^T \\ w_0 \end{pmatrix} = 0,$$

and

$$(-2((-y) - (-x)) - 2(-a)c_{\beta,1}(-y)^T J, c_{\beta,1}) \cdot (z_0, w_0) = 0.$$

We now see that (z_0, w_0) satisfies the system $S(-a, -y, -x)$. Since the above argument for proving nonexistence of nontrivial solution of $S(a, x, y)$ does not depend on specific values of a, x and y , the same conclusion holds for the system $S(-a, -y, -x)$. This completes the proof. \square

Remark 3.7. On \mathbb{R}^n , the oscillating kernel is of the form $|x|^{-\gamma} e^{i|x|^\beta}$ with $\beta \neq 0$. The behavior for the phases $|x|^\beta$ depends only on whether $\beta \neq 1$ or $\beta = 1$. Precisely, for $\beta \neq 1$, we have $\det \left(\frac{\partial^2}{\partial x \partial y} |x - y|^\beta \right) \neq 0$ for any (x, y) with $x \neq y$, but $\det \left(\frac{\partial^2}{\partial x \partial y} |x - y| \right) = 0$ for any (x, y) with $x \neq y$ and this case correspond to Bochner-Riesz means operators, which still remains as a conjecture. On hand, the phase $\rho((x, t) \cdot (y, s)^{-1})^\beta$ has fold of the highest order type when $\beta = 1$ or $\beta = 2$,

which also remains open in this paper. In order to establish the sharp L^2 estimate for these cases, we would need to improve the current theory of oscillatory integral estimates for degenerate phases to higher orders (see [2, 6, 7]).

Remark 3.8. We note that from Lemma 3.6 and Case 3 of Lemma 3.5,

$$(3.27) \quad \|L_{A_j}\|_{L^2 \rightarrow L^2} + \|L_{B_j}\|_{L^2 \rightarrow L^2} \lesssim 2^{j(\alpha-n\beta)}$$

holds for all cases. It will be sufficient to use this weaker bound for the Hardy spaces estimates in Section 5.

4. HARDY SPACES ON THE HEISENBERG GROUPS

In this section we recall some properties of Hardy spaces on the Heisenberg group. We refer Coifman-Weiss [1] and Folland-Stein [5] for the details. From now on, we shall write $\rho(x)$ (resp., $x \cdot y$) just as $|x|$ (resp., xy) for notational convenience. It is known that $|x \cdot y| \leq |x| + |y|$ holds for all $x, y \in \mathbb{H}_a^n$ (see [12, p. 688]).

The left-invariant vector fields on \mathbb{H}_a^n is spanned by $T = \frac{\partial}{\partial t}$ and $X_j = \frac{\partial}{\partial x_j} + 2ax_{n+j}\frac{\partial}{\partial t}$, $X_{j+n} = \frac{\partial}{\partial x_{j+n}} - 2ax_n\frac{\partial}{\partial t}$, $1 \leq j \leq n$. Let $Y_j = X_j$ for $1 \leq j \leq 2n$ and $Y_{2n+1} = T$. We say that the right-invariant differential operator $Y^I = Y_1^{i_1} \cdots Y_{2n+1}^{i_{2n+1}}$ has homogeneous degree $d(I) = i_1 + i_2 + \cdots + i_{2n} + 2i_{2n+1}$. For $a \in \bar{\mathbb{N}}$, we define \mathcal{P}_a to be the set of all homogeneous polynomials of degree a .

Suppose that $x \in \mathbb{H}_a^n$, $a \in \bar{\mathbb{N}}$, and f is a function whose distributional derivatives $Y^I f$ are continuous in a neighborhood of x for $d(I) \leq a$. The homogeneous right *Taylor polynomial* of f at x of degree a is the unique $P_{f,x} \in \mathcal{P}_a$ such that $Y^I P_{f,x}(0) = Y^I f(x)$ for $d(I) \leq a$.

Proposition 4.1 ([5]). *Suppose that $f \in C^{k+1}$, $T \in \mathcal{S}'$, and $P_{f,x}(y) = \sum_{d(I) \leq k} a_I(x) \eta^I(y)$ is the right Taylor polynomial of f at x of homogeneous degree k . Then a_I is a linear combination of the $Y^J f$ for $d(J) \leq k$,*

$$(4.1) \quad |f(yx) - P_{f,x}(y)| \leq C_k |y|^{k+1} \sup_{\substack{d(I)=k+1 \\ |z| \leq b^{k+1}|y|}} |Y^I f(zx)|.$$

We will use some properties for H^p functions including the atomic decomposition and the molecular characterization. For $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, $s \in \mathbb{Z}$ and $s \geq [(2n+2)(1/p-1)]$, we say that the triple (p, q, s) is admissible.

Definition 4.2. For an admissible triple (p, q, s) , we define (p, q, s) -atom centered at x_0 as a function $a \in L^q(\mathbb{H}^n)$ supported on a ball $B \subset \mathbb{H}_a^n$ with center x_0 in such way that

- (i) $\|a\|_q \leq |B|^{1/q-1/p}$.
- (ii) $\int_{\mathbb{H}^n} a(x) P(x) dx = 0$ for all $P \in \mathcal{P}_s$.

Later, we will choose $q = 2$ to use the L^2 boundedness (3.27) obtained in Section 3.

Proposition 4.3 (Atomic decomposition in \mathbb{H}^p ; see [1]). *Let (p, q, s) be an admissible triple. Then any f in H^p can be represented as a linear combination of (p, q, s) -atoms,*

$$f = \sum_{i=1}^{\infty} \lambda_i f_i, \quad \lambda_i \in \mathbb{C},$$

where the f_i are (p, q, s) -atoms and the sum converges in H^p . Moreover, $\|f\|_{H^p}^p \sim \inf \{ \sum_{i=1}^{\infty} |\lambda_i|^p : \sum \lambda_i f_i \text{ is a decomposition of } f \text{ into } (p, q, s)\text{-atoms} \}$.

For an admissible triple (p, q, s) , we choose an arbitrary real number $\epsilon > \max\{s/(2n+2), 1/p-1\}$. Then we call (p, q, s, ϵ) an admissible quadruple. Now we introduce the molecules.

Definition 4.4. Let (p, q, s, ϵ) be an admissible quadruple. We set

$$(4.2) \quad a = 1 - 1/p + \epsilon, \quad b = 1 - 1/q + \epsilon.$$

A (p, q, s, ϵ) -molecule centered at x_0 is a function $M \in L^q(\mathbb{H}^n)$ such that

- (1) $M(x) \cdot |x_0^{-1}x|^{(2n+2)b} \in L^q(\mathbb{H}^n)$.
- (2) $\|M\|_q^{a/b} \cdot \|M(x) \cdot |x_0^{-1}x|^{(2n+2)b}\|_q^{1-a/b} \equiv \mathcal{N}(M) < \infty$.
- (3) $\int_{\mathbb{H}^n} M(x)P(x)dx = 0$ for every $P \in \mathcal{P}_s$.

Theorem 4.5.

- (1) Every (p, q, s') -atom f is a (p, q, s, ϵ) -molecule for any $\epsilon > \max\{s/(2n+2), 1/p-1\}$, $s \leq s'$ and $\mathcal{N}(f) \leq C_1$, where the constant C_1 is independent of the atom.
- (2) Every (p, q, s, ϵ) -molecule M is in H^p and $\|M\|_{H^p} \leq C_2 \mathcal{N}(M)$, where the constant C_2 is independent of the molecule.

Thanks to this Theorem, in order to verify that T is bounded on H^p it is enough to show that, for all p -atoms f , the function Tf is a p -molecule and $\mathcal{N}(Tf) \leq C$ for some constant C independent of f .

5. H^p ESTIMATES

We start with a lemma which will be useful in the proofs of the sequel.

Lemma 5.1.

- (1) Suppose that $d < 0$, $c + d < 0$ and $B > 1$. Then

$$\sum_{j=1}^{\infty} 2^{cj} \min\{1, B2^{dj}\} \lesssim 1 + (\log B)B^{-\frac{c}{d}}.$$

- (2) Suppose that $c < 0$, $d > 0$ and $B < 1$. Then

$$\sum_{j=1}^{\infty} 2^{cj} \min\{1, B2^{dj}\} \lesssim B + |\log B|B^{-\frac{c}{d}}.$$

Proof. Set $K = \sum_{j=1}^{\infty} 2^{cj} \min\{1, B2^{dj}\}$. Then,

$$K = \sum_{B2^{dj} \leq 1} 2^{(c+d)j} + \sum_{B2^{dj} > 1} 2^{cj}.$$

A straightforward calculation gives the bound for K . Suppose that $d < 0, c + d > 0$ and $B > 1$. Then

- $K \lesssim 1$ for $c < 0$,
- $K \lesssim \log B$ for $c = 0$,
- $K \lesssim B^{-\frac{c}{d}}$ for $c > 0$.

In any case we see that $K \lesssim 1 + (\log B)B^{-\frac{c}{d}}$. Suppose now that $c < 0, d > 0$ and $B < 1$. Then

- $K \lesssim B$ for $c + d < 0$,
- $K \lesssim \log B \cdot B$ for $c + d = 0$,
- $K \lesssim B^{-\frac{c}{d}}$ for $c + d > 0$.

In any case we have $K \lesssim B + |\log B|B^{-\frac{\epsilon}{2}}$. The Lemma is proved. \square

Theorem 5.2. Assume $p \leq 1$ and $(\frac{1}{p} - 1)(2n + 2)\beta + \alpha < 0$. Then $T_{K_{\alpha,\beta}}$ is bounded on H^p .

Proof. From the decomposition of kernel (2.1), we have

$$\|K_{\alpha,\beta} * f\|_{H^p}^p \leq \sum_{j \geq 1} \|K_{\alpha,\beta}^j * f\|_{H^p}^p.$$

We shall bound the norm $\|K_{\alpha,\beta}^j * f\|_{H^p}$ for each $j \in \mathbb{N}$ by some constant multiple of $\|f\|_{H^p}$. Notice that $K_j(x, t) = \rho(x, t)^{-(2n+2+\alpha)} e^{i\rho(x, t)^{-\beta}} \chi(2^j \rho(x, t))$. From the atomic decomposition for H^p space, it is enough to establish the estimate for any atom f supported on $B(0, R)$ with some $R > 0$ such that

$$(5.1) \quad \begin{aligned} & - \|f\|_{L^2} \leq R^{(2n+2)(\frac{1}{2} - \frac{1}{p})}, \\ & - \int f(x) x^\alpha dx = 0, \quad \text{for all } |\alpha| \leq s = [(2n+2)(\frac{1}{p} - 1)]. \end{aligned}$$

In view of part (2) of Theorem 4.5 it suffices to bound $\mathcal{N}(K_j * f)$. For an admissible quadruple, we choose an $\epsilon > \max\{\frac{s}{2n+2}, \frac{1}{p} - 1\} = \frac{1}{p} - 1$ and set $\epsilon = \frac{1}{p} - 1 + \delta$ with some $\delta > 0$. Then we have $a = \delta$ and $b = \frac{1}{p} - \frac{1}{2} + \delta$ in (4.2). We will choose δ sufficiently small later. Recall that $\mathcal{N}(K_j * f) = \|K_j * f\|_2^{a/b} \cdot \|K_j * f(x) \cdot |x|^{(2n+2)b}\|_2^{1-a/b}$. From the L^2 estimate (3.27) we get

$$(5.2) \quad \|K_j * f\|_2 \lesssim 2^{j(\alpha - n\beta)} \|f\|_2.$$

We have

$$\|K_j * f(x) \cdot |x|^{(2n+2)b}\|_2^2 = \int_{\mathbb{H}^n} |K_j * f(x)|^2 \cdot |x|^{2(2n+2)b} dx = I_1 + I_2,$$

where

$$I_1 = \int_{|x| \leq 2R} |K_j * f(x)|^2 \cdot |x|^{2(2n+2)b} dx \quad \text{and} \quad I_2 = \int_{|x| > 2R} |K_j * f(x)|^2 \cdot |x|^{2(2n+2)b} dx.$$

Then

$$(5.3) \quad \begin{aligned} \sum_{j \geq 1} \|K_j * f\|_{H^p}^p & \lesssim \sum_{j \geq 1} \mathcal{N}(K_j * f)^p \\ & \lesssim \sum_{j \geq 1} \left(\|K_j * f\|_2^{a/b} \cdot (I_1^{1/2(1-a/b)} + I_2^{1/2(1-a/b)}) \right)^p \\ & \lesssim \sum_{j \geq 1} \|K_j * f\|_2^{pa/b} \cdot I_1^{p/2(1-a/b)} + \sum_{j \geq 1} \|K_j * f\|_2^{pa/b} \cdot I_2^{p/2(1-a/b)} \end{aligned}$$

Set $S_1 = \sum_{j \geq 1} \|K_j * f\|_2^{pa/b} \cdot I_1^{p/2(1-a/b)}$ and $S_2 = \sum_{j \geq 1} \|K_j * f\|_2^{pa/b} \cdot I_2^{p/2(1-a/b)}$. Then it is enough to show that $S_1 \lesssim 1$ and $S_2 \lesssim 1$. We use (3.27) and (5.1) to bound I_1 as follows.

$$(5.4) \quad \begin{aligned} I_1 & \lesssim \int_{\mathbb{H}^n} |f * K_j(x)|^2 dx \cdot R^{2(2n+2)b} \lesssim 2^{2j(\alpha - n\beta)} \|f\|_2^2 \cdot R^{2(2n+2)b} \\ & \lesssim 2^{2j(\alpha - n\beta)} R^{2(2n+2)b} \cdot R^{(2n+2)(1-2/p)} \\ & \lesssim 2^{2j(\alpha - (n+1/2)\beta)} R^{2(2n+2)\delta}, \end{aligned}$$

where the last inequality comes from (5.1). From (5.2) and (5.4) we have

$$(5.5) \quad \begin{aligned} \|K_j * f\|_2^{a/b} \cdot I_1^{1/2(1-a/b)} & \lesssim \left\{ 2^{j(\alpha - n\beta)} R^{(2n+2)(1/2-1/p)} \right\}^{a/b} \cdot \left\{ 2^{j(\alpha - n\beta)} \cdot R^{(2n+2)\delta} \right\}^{(1-a/b)} \\ & = 2^{j(\alpha - n\beta)}, \end{aligned}$$

where the equality comes from the calculation $(\frac{1}{2} - \frac{1}{p})\frac{a}{b} + a(1 - \frac{a}{b}) = \frac{a}{b}(\frac{1}{2} - \frac{1}{p} - a) + a = \frac{a}{b}(-b) + a = 0$. Thus we have $S_1 \lesssim \sum_{j \geq 1} 2^{j(\alpha - n\beta)p} \lesssim 1$.

Now we consider I_2 and S_2 . We have $I_2 = 0$ for $R > 1$ since the support of $K_j * f$ is contained in the subset $\{x : |x| \leq 1 + R\}$ which is a subset of $\{x : |x| < 2R\}$ for $R > 1$. Thus we may only consider the case $R \leq 1$. In the following integral expression

$$(K_j * f)(x) = \int K_j(xy^{-1})f(y)dy,$$

We have $|xy^{-1}| \leq 2^{-j}$ and $|y| \leq R$. These imply $|x| \leq |xy^{-1}| + |y| \leq 2^{-j} + R$. It means that $I_2 = 0$ for $2^{-j} < R$. Thus we only need to consider $j \in \mathbb{N}$ such that $2^{-j} \geq R$, for which we have $|x| \leq 2^{-j+1}$ for $x \in \text{Supp}(K_j * f)$. Then we get

$$(5.6) \quad I_2 = \int_{|x| > 2R} |f * K_j(x)|^2 \cdot |x|^{2(2n+2)b} dx \lesssim \int_{|x| > 2R} |f * K_j(x)|^2 dx \cdot 2^{-2(2n+2)bj}.$$

From Proposition 4.1, for any $I \in \mathbb{N}_0$, there is a polynomial P_j^x of degree $\leq I$ such that

$$(5.7) \quad \begin{aligned} |K_j(xy^{-1}) - P_j^x(y)| &\lesssim |y|^{I+1} \sup_{|\alpha| \leq I+1} |X^\alpha K_j(xy^{-1})| \\ &\lesssim |y|^{I+1} 2^{j(2n+2+\alpha)} 2^{j(\beta+1)(I+1)}. \end{aligned}$$

From (5.1) we get the identity for $0 \leq I \leq s$,

$$K_j * f(x) = \int (K_j(xy^{-1}) - P_j^x(y))f(y)dy.$$

Note that $f(y)$ has support in $|y| \leq R$, then from (5.1) and (5.7) we get

$$(5.8) \quad |K_j * f(x)| \lesssim R^{I+1} 2^{j(2n+2+\alpha)} 2^{j(\beta+1)(I+1)} \int_{|y| \leq R} |f(y)| dy$$

$$(5.9) \quad \begin{aligned} &\lesssim R^{I+1} 2^{j(2n+2+\alpha)} 2^{j(\beta+1)(I+1)} R^{\frac{1}{2}(2n+2)} \|f\|_2 \\ &\lesssim 2^{j(2n+2+\alpha)} (R 2^{j(\beta+1)})^{(I+1)} R^{(2n+2)(1-\frac{1}{p})}. \end{aligned}$$

Now we can estimate (5.6) as

$$(5.10) \quad \begin{aligned} I_2 &\lesssim 2^{-2(2n+2)bj} 2^{-j(2n+2)} \left\{ 2^{j(2n+2+\alpha)} (R 2^{j(\beta+1)})^{(I+1)} R^{(2n+2)(1-\frac{1}{p})} \right\}^2 \\ &= 2^{2j\{(2n+2)(1-\frac{1}{p}-\delta)+\alpha\}} (R 2^{j(\beta+1)})^{2(I+1)} R^{2(2n+2)(1-\frac{1}{p})}. \end{aligned}$$

Here we may choose $I = 0$ or $I = s$, which gives

$$I_2 \lesssim 2^{2j\{(2n+2)(1-\frac{1}{p}-\delta)+\alpha\}} R^{2(2n+2)(1-\frac{1}{p})} \min\{1, (R 2^{j(\beta+1)})^{2(s+1)}\}.$$

Now we have

$$(5.11) \quad \begin{aligned} \|K_j * f\|_2^{a/b} \cdot I_2^{\frac{1}{2}(1-a/b)} &\lesssim \{2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)}\}^{a/b} \\ &\cdot \{2^{j\{(2n+2)(1-\frac{1}{p}-\delta)+\alpha\}} R^{(2n+2)(1-\frac{1}{p})} \min\left(1, (R 2^{j(\beta+1)})^{(s+1)}\right)\}^{(1-a/b)}. \end{aligned}$$

From $p \leq 1$ and $\alpha < 0$ we have $(2n+2)(1-\frac{1}{p}-\delta)+\alpha < 0$. Thus, if $\min(1, (R 2^{j(\beta+1)})^{s+1}) = 1$ the exponent of 2^j is smaller than zero provided a is small enough. Recall that $R \leq 1$. Then, using (2) in Lemma 5.1 we get

$$\sum_{j \geq 1} \|K_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim R^{p\mu_\delta} + |\log R| \cdot R^{p\kappa_\delta},$$

where

$$R^{(2n+2)(1/2-1/p)} \cdot (R^{(2n+2)(1-\frac{1}{p})+(s+1)p(1-a/b)})^{\frac{1}{p}} \cdot \left[R^{-\frac{1}{\beta+1}[\alpha-(n+1/2)\beta]} R^{(2n+2)(1/2-1/p)} \right]^{p\delta/b} \cdot \left[R^{-\frac{1}{\beta+1}[(2n+2)(1-1/p-\delta)+\alpha]} R^{(2n+2)(1-1/p)} \right]^{p(1-\delta/b)}.$$

Observe that

$$\mu_0 = \{(2n+2)(1-\frac{1}{p}) + (s+1)\} > 0,$$

and

$$\kappa_0 = -\frac{1}{\beta+1}[(2n+2)\beta(\frac{1}{p}-1) + \alpha] > 0.$$

Thus, for δ small enough, we have $\mu_\delta, \kappa_\delta > 0$ and since $R \leq 1$,

$$(5.13) \quad \sum_{j \geq 1} \|K_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim R^{p\mu_\delta} + |\log R| \cdot R^{p\kappa_\delta} \leq 1.$$

We then conclude that $S_2 \lesssim 1$. The proof is complete. \square

We now consider $T_{L_{\alpha,\beta}}$. Observe that the oscillating term $e^{i\rho(x,t)^\beta}$ exhibits different behavior whether $0 < \beta < 1$ or $\beta > 1$. As ρ goes to infinity, the oscillation becomes faint if for the case $0 < \beta < 1$. In contrary, the oscillation grows to infinity for $\beta > 1$. Hence we deal with the two cases separately.

Theorem 5.3. *Assume $0 < \beta < 1$ and $p \leq 1$ and $(\frac{1}{p}-1)(2n+2)\beta + \alpha < 0$. Then the operator $T_{L_{\alpha,\beta}}$ is bounded on H^p space.*

Proof. From (2.2) we have

$$(5.14) \quad \|L_{\alpha,\beta} * f\|_{H^p}^p \leq \sum_{j \geq 1} \|L_{\alpha,\beta}^j * f\|_{H^p}^p.$$

We now estimate each norm $\|L_{\alpha,\beta}^j * f\|_{H^p}$ by $\|f\|_{H^p}$. From the atomic decomposition for H^p space, we may choose f as an atom supported on $B(0, R)$ with some $R > 0$, which satisfies

$$(5.15) \quad \begin{aligned} & - \|f\|_{L^2} \leq R^{(2n+2)(\frac{1}{2}-\frac{1}{p})}, \\ & - \int f(x) x^\alpha dx = 0, \quad \text{for all } |\alpha| \leq s = [(2n+2)(\frac{1}{p}-1)]. \end{aligned}$$

From (b) in Theorem 4.5, it suffices to estimate $\mathcal{N}(L_j * f)$. For an admissible quadruple (p, q, s, ϵ) we may choose any $\epsilon > \max\{\frac{s}{2n+2}, \frac{1}{p}-1\} = \frac{1}{p}-1$. Simply we let $\epsilon = \frac{1}{p}-1 + \delta$ with some $\delta > 0$. Then we have $a = \delta$ and $b = \frac{1}{p}-\frac{1}{2} + \delta$. for (4.2). We will choose δ sufficiently small later.

From (3.27) we have

$$\|L_j * f\|_2 \lesssim 2^{j(\alpha-n\beta)} \|f\|_2.$$

We have

$$(5.16) \quad \|L_j * f(x) \cdot |x|^{(2n+2)b}\|_2^2 = \int_{\mathbb{H}^n} |L_j * f(x)|^2 \cdot |x|^{2(2n+2)b} dx = I_1 + I_2,$$

where

$$I_1 = \int_{|x| \leq 2R} |L_j * f(x)|^2 \cdot |x|^{2(2n+2)b} dx \quad \text{and} \quad I_2 = \int_{|x| > 2R} |L_j * f(x)|^2 \cdot |x|^{2(2n+2)b} dx.$$

Then,

$$\begin{aligned}
(5.17) \quad \sum_{j \geq 1} \|L_j * f\|_{H^p}^p &\lesssim \sum_{j \geq 1} \mathcal{N}(L_j * f)^p \\
&\lesssim \sum_{j \geq 1} \left(\|L_j * f\|_2^{a/b} \cdot (I_1^{1/2(1-a/b)} + I_2^{1/2(1-a/b)}) \right)^p \\
&\lesssim \sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_1^{p/2(1-a/b)} + \sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{p/2(1-a/b)}
\end{aligned}$$

Set $S_1 = \sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_1^{p/2(1-a/b)}$ and $S_2 = \sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{p/2(1-a/b)}$. Then it is enough to show that $S_1 \lesssim 1$ and $S_2 \lesssim 1$. First we estimate I_1 with L^2 estimates (3.27) as follows

$$\begin{aligned}
(5.18) \quad I_1 &\lesssim \int_{\mathbb{H}^n} |f * L_j(x)|^2 dx \cdot R^{2(2n+2)b} \lesssim 2^{2j(\alpha-n\beta)} \|f\|_2^2 \cdot R^{2(2n+2)b} \\
&\leq 2^{2j(\alpha-n\beta)} R^{2(2n+2)b} \cdot R^{(2n+2)(1-2/p)} = 2^{2j(\alpha-(n+1/2)\beta)} R^{2(2n+2)\delta}.
\end{aligned}$$

Thus we can bound $\|L_j * f\|_2^{a/b} \cdot I_1^{\frac{1}{2}(1-a/b)}$ as

$$\begin{aligned}
(5.19) \quad \|L_j * f\|_2^{a/b} \cdot I_1^{1/2(1-a/b)} &\lesssim \left\{ 2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)} \right\}^{a/b} \cdot \left\{ 2^{j(\alpha-n\beta)} \cdot R^{(2n+2)\delta} \right\}^{(1-a/b)} \\
&= 2^{j(\alpha-n\beta)},
\end{aligned}$$

and we have $S_1 \lesssim \sum_{j \geq 1} 2^{j(\alpha-n\beta)p} \lesssim 1$.

For I_2 we consider the two cases $R > 1$ and $R \leq 1$.

Case (i): Suppose $R > 1$. In the integral

$$(L_j * f)(x) = \int L_j(xy^{-1})f(y)dy,$$

we have $|xy^{-1}| \leq 2^j$ and $|y| \leq R$, which imply $|x| \leq |xy^{-1}| + |y| \leq 2^j + R$. Therefore, in (5.16), we have that $I_2 = 0$ for $2^j < R$. Thus we only need to consider j with $2^j \geq R$. Then we have $|x| \leq 2^{j+1}$ for x in the support of $L_j * f$, and so

$$(5.20) \quad I_2 \lesssim \int_{|x| > 2R} |f * L_j(x)|^2 dx \cdot 2^{2(2n+2)bj}.$$

By (4.1) we have

$$\begin{aligned}
(5.21) \quad |L_j(xy^{-1}) - P_j^x(y)| &\lesssim |y|^{I+1} \sup_{|\alpha| \leq I+1} |X^\alpha L_j(xy^{-1})| \\
&\lesssim |y|^{I+1} 2^{-j(2n+2-\alpha)} 2^{j(\beta-1)(I+1)}.
\end{aligned}$$

Since $f(y)$ has support in $|y| \leq R$ and (5.15), we have

$$(5.22) \quad |L_j * f(x)| \lesssim R^{I+1} 2^{-j(2n+2-\alpha)} 2^{j(\beta-1)(I+1)} \int_{|y| \leq R} |f(y)| dy$$

$$\begin{aligned}
(5.23) \quad &\lesssim R^{I+1} 2^{-j(2n+2-\alpha)} 2^{-j(\beta-1)(I+1)} R^{\frac{1}{2}(2n+2)} \|f\|_2 \\
&\lesssim 2^{-j(2n+2-\alpha)} (R 2^{-j(\beta-1)})^{(I+1)} R^{(2n+2)(1-\frac{1}{p})}.
\end{aligned}$$

Thus we can estimate (5.20) as

$$\begin{aligned}
(5.24) \quad I_2 &\lesssim 2^{2(2n+2)bj} 2^{j(2n+2)} \left\{ 2^{-j(2n+2-\alpha)} (R 2^{-j(\beta-1)})^{(I+1)} R^{(2n+2)(1-\frac{1}{p})} \right\}^2 \\
&= 2^{2j\{(2n+2)(1/p+1+\delta)+\alpha\}} (R 2^{j(\beta-1)})^{2(I+1)} R^{2(2n+2)(1-\frac{1}{p})}.
\end{aligned}$$

Here we may choose $I = 0$ and $I = s$, which gives

$$I_2 \lesssim 2^{2j\{(2n+2)(1/p-1+\delta)+\alpha\}} R^{2(2n+2)(1-\frac{1}{p})} \min\{1, (R2^{j(\beta-1)})^{2(s+1)}\}.$$

Thus,

$$(5.25) \quad \|L_j * f\|_2^{a/b} \cdot I_2^{\frac{1}{2}(1-a/b)} \lesssim \{2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)}\}^{a/b} \cdot \{2^{j\{(2n+2)(1/p-1+\delta)+\alpha\}} R^{(2n+2)(1-\frac{1}{p})} \min\left(1, (R2^{j(\beta-1)})^{(s+1)}\right)\}^{(1-a/b)}.$$

Provided δ is small enough, we have

$$(5.27) \quad \begin{aligned} (2n+2)\left(\frac{1}{p} - 1 + \delta\right) + \alpha + (\beta - 1)(s + 1) &= (2n + 2)\left(\frac{1}{p} - 1 + \delta\right) + \alpha + (\beta - 1)\left[\left((2n + 2)\left(\frac{1}{p} - 1\right)\right) + 1\right] \\ &< (2n + 2)\left(\frac{1}{p} - 1 + \delta\right) + \alpha + (\beta - 1)(2n + 2)\left(\frac{1}{p} - 1\right) \\ &= (2n + 2)\left(\frac{1}{p} - 1\right)\beta + \alpha + (2n + 2)\delta < 0. \end{aligned}$$

Therefore the index of 2^j in (5.25) with $(R2^{j(\beta-1)})^{s+1}$ is negative for small $\delta > 0$. Remind that $R > 1$. Then, from (1) in Lemma 5.1 we have

$$\sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim R^{p\mu_\delta} + \log(R + 1)R^{p\kappa_\delta},$$

where

$$(5.28) \quad \begin{aligned} R^{p\mu_\delta} &= R^{(2n+2)(1/2-1/p)\frac{pa}{b} + (2n+2)(1-1/p)p(1-a/b)}, \\ R^{p\kappa_\delta} &= [R^{-\frac{1}{1-\beta}[\alpha-n\beta]} R^{(2n+2)(1/2-1/p)p\delta/b} \cdot [R^{\frac{1}{1-\beta}\{(2n+2)(1/p-1+\delta)+2\alpha\}} \cdot R^{(2n+2)(1-1/p)p(1-a/b)}]. \end{aligned}$$

Because $p \leq 1$, we easily see that $\mu_\delta \leq 0$. Moreover,

$$\kappa_0 = \frac{1}{1-\beta} \{\beta(2n+2)\left(\frac{1}{p} - 1\right) + \alpha\} < 0.$$

From this, we get $\kappa_\delta < 0$ for δ small enough. Therefore we have

$$S_2 \lesssim R^{\mu_\delta} + \log(R + 1)R^{\kappa_\delta} \lesssim 1.$$

Case (ii): Suppose $R \leq 1$. We see that $\min(1, (R2^{j(\beta-1)})^{(s+1)}) = R2^{j(\beta-1)(s+1)}$ and (5.25) becomes

$$(5.29) \quad \sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim \{2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)}\}^{pa/b} \cdot \{2^{j(2n+2)(1/p-1+\delta)+\alpha} R^{(2n+2)(1-\frac{1}{p})} \cdot (R2^{j(\beta-1)})^{(s+1)}\}^{p(1-a/b)}.$$

Because the power of 2^j is negative, provided δ is small enough, we get

$$(5.30) \quad \begin{aligned} \sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} &\lesssim R^{(2n+2)(1/2-1/p)\frac{pa}{b}} \cdot R^{\{(2n+2)(1-\frac{1}{p})+(s+1)p(1-\frac{a}{b})\}} \\ &=: R^{p\mu_\delta}. \end{aligned}$$

Observe that

$$\mu_0 = (2n + 2)\left(1 - \frac{1}{p}\right) + (s + 1) = (2n + 2)\left(1 - \frac{1}{p}\right) + \left[\left((2n + 2)\left(\frac{1}{p} - 1\right)\right) + 1\right] > 0.$$

Thus we have $\mu_\delta > 0$ for δ small enough. Now we get

$$\sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim R^{p\mu_\delta} \leq 1.$$

We then conclude that $S_2 \lesssim 1$. The proof is complete. \square

We now establish the same result for the case $\beta > 1$.

Theorem 5.4. *For $1 < \beta$, $p \leq 1$, if $(\frac{1}{p} - 1)(2n + 2)\beta + \alpha < 0$, the operator $T_{L_{\alpha, \beta}}$ is bounded on H^p space.*

Proof. By arguing as in (5.14)–(5.17) in the proof of Theorem 5.3 to obtain the following

$$(5.31) \quad \sum_{j \geq 1} \|L_j * f\|_{H^p}^p \lesssim \sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_1^{p/2(1-a/b)} + \sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{p/2(1-a/b)},$$

where I_1 and I_2 are defined as in (5.16). Because the estimate for I_1 is exactly same with the proof of Theorem 5.3, we only deal with I_2 . As before, we have

$$(5.32) \quad \begin{aligned} \|L_j * f\|_2^{a/b} \cdot I_2^{\frac{1}{2}(1-a/b)} &\lesssim \{2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)}\}^{a/b} \\ &\cdot \{2^{j\{(2n+2)(1/p-1+\delta)+\alpha\}} R^{(2n+2)(1-\frac{1}{p})} \min\left(1, (R2^{j(\beta-1)})^{(s+1)}\right)\}^{(1-a/b)} \end{aligned}$$

Case (i): Suppose $R > 1$. As for the case $\beta < 1$, we have $I_2 = 0$ if $2^j < R$ and we only need consider j with $2^j \geq R$. Since $R2^{j(\beta-1)} \geq 1$, we estimate I_2 as

$$I_2 \lesssim 2^{j\{2(2n+2)[1/p-1+\delta]+2\alpha\}} R^{2(2n+2)(1-1/p)}.$$

Note that

$$(5.33) \quad (2n+2)\left(\frac{1}{p} - 1\right) + \alpha < (2n+2)\left(\frac{1}{p} - 1\right)\beta + \alpha < 0.$$

Thus, if δ is sufficiently small, we have $(2n+2)(1/p-1+\delta) + \alpha < 0$ and we can sum (5.32) as

$$(5.34) \quad \sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim R^{(2n+2)(1/2-1/p)\frac{pa}{b}} \cdot R^{\{(2n+2)(1-\frac{1}{p})\}p(1-\frac{a}{b})} \leq 1,$$

where the last inequality holds because $p \leq 1$ and $R > 1$.

Case (ii): Suppose $R \leq 1$. From (5.33), using (1) in Lemma 5.1 we have

$$\sum_{j \geq 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{\frac{1}{2}p(1-a/b)} \lesssim R^{p\mu_\delta} + |\log R| R^{p\kappa_\delta},$$

where

$$(5.35) \quad \begin{aligned} R^{\mu_\delta} &= R^{(2n+2)(1/2-1/p)\frac{a}{b}} \cdot R^{\{(2n+2)(1-1/p)+(s+1)\}(1-\frac{a}{b})}, \\ R^{\kappa_\delta} &= R^{(2n+2)(1/2-1/p)\frac{a}{b}} \cdot \{R^{\frac{1}{1-\beta}\{(2n+2)(1/p-1+\delta)+\alpha\}} R^{(2n+2)(1-1/p)}\}^{1-a/b}. \end{aligned}$$

Observe that

$$\mu_0 = (2n+2)\left(1 - \frac{1}{p}\right) + (s+1) = (2n+2)\left(1 - \frac{1}{p}\right) + [(2n+2)\left(\frac{1}{p} - 1\right)] + 1 > 0$$

and

$$\kappa_0 = \frac{1}{1-\beta} \{(2n+2)(1/p-1) + \alpha\} + (2n+2)(1-1/p) = \frac{1}{1-\beta} \{\beta(2n+2)\left(\frac{1}{p} - 1\right) + \alpha\} > 0.$$

Therefore we have $\mu_\delta, \kappa_\delta > 0$ for δ small enough, and so

$$(5.36) \quad \sum_{j \geq 1} \|L_j * f\|_2^{a/b} \cdot I_2^{\frac{1}{2}(1-a/b)} \lesssim R^{\mu_\delta} + |\log R| \cdot R^{\kappa_\delta} \leq 1.$$

Now we conclude that $S_2 \lesssim 1$ from (5.34) and (5.36). The proof is complete. \square

6. NECESSARY CONDITIONS

In this section we show that the Hardy space boundedness obtained in the previous section is sharp except for the endpoint cases. We only give an example for Theorem 5.2. Examples for the other theorems can be found similarly. We refer to Sjölin [19] for the Euclidean case.

We let $g(x)$ a function such that

$$\int_{\mathbb{R}} x^\alpha g(x) dx = 0 \quad \text{for } 0 \leq \alpha \leq k \quad \text{and} \quad \int_{\mathbb{R}} x^{k+1} g(x) dx \neq 0.$$

Let $h(x_2, \dots, x_{2n}, x_{2n+1})$ a function supported on the ball $B(0, 1)$ such that $\int_{\mathbb{R}^{2n}} h \neq 0$ and let f be the function on \mathbb{R}^{2n+1} defined by $f(x_1, \dots, x_{2n+1}) = g(x_1)h(x_2, \dots, x_{2n+1}) \forall (x_1, \dots, x_{2n+1}) \in \mathbb{R}^{2n+1}$. Then

$$\int_{\mathbb{H}^n} x^\alpha f(x) = 0, \quad \text{if } |\alpha| \leq k.$$

For $\epsilon > 0$ set $f_\epsilon(x) = \epsilon^{-(2n+2)/p} f(\frac{x}{\epsilon})$. We note that $\|f_\epsilon\|_{H^p} = C$ for all $\epsilon > 0$. Assume that $T_{K_{\alpha,\beta}}$ is bounded on H^p . Then $\|T_{K_{\alpha,\beta}}(f_\epsilon)\|_{H^p} \lesssim 1$. Note that $|y| \leq \epsilon$ for $y \in \text{supp}(f_\epsilon)$. Then, for $|x| \geq C\epsilon$ with a large constant $C > 0$, we have

$$\begin{aligned} (6.1) \quad K(x)f(x) &= \int K(xy^{-1})f_\epsilon(y)dy \\ (6.2) \quad &= \int \left(K(xy^{-1}) - \sum_{|\alpha| \leq k+1} \frac{1}{\alpha!} D^\alpha K(x)y^\alpha \right) f_\epsilon(y)dy + \int \left(\sum_{|\alpha| \leq k+1} \frac{1}{\alpha!} D^\alpha K(x)y^\alpha \right) f_\epsilon(y)dy \end{aligned}$$

$$\begin{aligned} (6.3) \quad &= \int D^{k+2}K(xy_*^{-1})O(y^{k+2})f_\epsilon(y)dy + C\partial_{x_1}^{k+1}K(x) \int_{\mathbb{R}} y_1^{k+1}f_\epsilon(y_1)dy_1, \quad |y_*| \leq |y| \leq \epsilon \\ &= O(\epsilon^{(2n+2)+k+2-\frac{(2n+2)}{p}}|x|^{-(n+\alpha+(k+2)(\beta+1))}) + \epsilon^{k+1+(2n+2)-\frac{(2n+2)}{p}}\partial_{x_1}^{k+1}K(x). \end{aligned}$$

Take $K(x) = |x|^{-2n-2-\alpha} e^{i|x|^{-\beta}} \chi(x)$. We see that $|\partial_{x_1}^{k+1}K(x)| \sim |x|^{-(2n+2)-\alpha-(k+1)(\beta+1)}$ for small x . For $\epsilon \lesssim |x|^{\beta+1}$ we have

$$\epsilon^{(2n+2)+k+2-\frac{(2n+2)}{p}}|x|^{-(2n+2+\alpha+(k+2)(\beta+1))} \lesssim \epsilon^{(2n+2)+(k+1)-\frac{(2n+2)}{p}}|x|^{-(2n+2)-\alpha-(k+1)(\beta+1)}.$$

Therefore we get

$$K_{\alpha,\beta} * f_\epsilon(x) \sim \epsilon^{(2n+2)+k+1-\frac{(2n+2)}{p}}|x|^{-(2n+2)-\alpha-(k+1)(\beta+1)} \quad \text{for } |x| \gtrsim \epsilon^{1/(\beta+1)}.$$

Then,

$$(6.4) \quad \int_{\mathbb{H}^n} |K_{\alpha,\beta} * f_\epsilon(x)|^p dx \gtrsim \epsilon^{p(2n+2)+kp+p-(2n+2)} \int_{c \geq |x| \gtrsim \epsilon^{1/(\beta+1)}} |x|^{-(2n+2)p-\alpha p-(k+1)(\beta+1)p} dx$$

$$\begin{aligned} (6.5) \quad &\gtrsim \epsilon^{p(2n+2)+kp+p-(2n+2)} \epsilon^{-\frac{(2n+2)p-(2n+2)+\alpha p}{\beta+1}-(k+1)p} \\ &= \epsilon^{\frac{-p}{\beta+1}[(\frac{1}{p}-1)(2n+2)\beta+\alpha]}. \end{aligned}$$

This implies that $(1 - \frac{1}{p})(2n+2)\beta + \alpha$ must be ≤ 0 . This shows that Theorem 5.2 is sharp except the endpoint case $(1 - \frac{1}{p})(2n+2)\beta + \alpha = 0$.

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